Three-dimensional uniform spanning trees

Saraí Hernández-Torres
Technion

Joint work with
Omer Angel (UBC), David Croydon (Kyoto),
Daisuke Shiraishi (Kyoto)
1. Uniform spanning trees (UST)
2. Scaling limits of uniform spanning trees on $\mathbb{Z}^d$
3. Loop-erased random walks
4. The intrinsic ball 🙌
Uniform spanning trees
Uniform spanning tree (UST)

- $G=(V,E)$: finite and connected graph.
- Spanning tree of $G$: subgraph of $G$ spanning all the vertices without cycles.
- Uniform spanning tree of $G$: uniform sample over the collection of spanning trees.
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$\Lambda_n \uparrow \mathbb{Z}^d$
Uniform spanning forest (USF)

- The uniform spanning forest measure on $\mathbb{Z}^d$ is the weak limit:

$$P(B \subset \text{USF}(\mathbb{Z}^d)) = \lim_{n \to \infty} P(B \subset \text{UST}(\Lambda_n)),$$

where $B$ is a finite set of edges.
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[PEmantle, ’91]

- The USF of $\mathbb{Z}^d$ exists and it is unique.
- The USF of $\mathbb{Z}^d$ is a tree (and call it UST) if, and only if, $d \leq 4$. 
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This talk: $d = 2, 3$
UST as a measured, rooted, spatial tree

- $\mathcal{U}$: UST of $\mathbb{Z}^d$
- $d$: intrinsic metric on $\mathcal{U}$
- $\mu$: counting measure on $\mathcal{U}$
- $\phi : \mathcal{U} \to \mathbb{R}^d$ identity
- $\rho = 0$: root

$(\mathcal{U}, d)$ is a real tree, and

$$\mathcal{U} = (\mathcal{U}, d, \mu, \phi, \rho)$$

is a measured, rooted, spatial tree.
**Measured, rooted, spatial trees**

\( \mathcal{T} \) is the space of measured, rooted, spatial trees:

\[
\mathcal{T} = (\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}})
\]

where

- \((\mathcal{T}, d_{\mathcal{T}})\): complete and locally compact real tree,
- \(\mu_{\mathcal{T}}\): locally finite Borel measure,
- \(\phi_{\mathcal{T}} : \mathcal{T} \to \mathbb{R}^d\): continuous map,
- \(\rho_{\mathcal{T}}\): root.

\(\mathcal{T}\) is a separable metrizable space.

[Abraham-Delmas-Hoscheit '13], [Barlow-Croydon-Kumagai, '17]
Scaling limits of uniform spanning trees on $\mathbb{Z}^d$
Scaling of the UST

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- $d$: intrinsic metric on $\mathcal{U}$
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$\underline{\mathcal{U}} = (\mathcal{U}, \delta^d, \delta^d\mu, \delta^d\phi, \rho)$,

where $\beta$ is the growth exponent of the path between two vertices of the UST.

$(\underline{\mathcal{U}}_\delta)_{\delta \in (0,1)}$
Scaling of the UST

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\mathcal{U}_\delta = (\mathcal{U}, \delta^\beta d, \delta^d \mu, \delta \phi, \rho),
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where \( \beta \) is the growth exponent of the loop-erased random walk.

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Scaling limit of USTs in 2D

Let $P_\delta$ be the law of the measured spatial space $(T, \delta^{5/4} d, \delta^2 \mu, \delta \phi, \rho)$,

Then

$(P_\delta)_{\delta \in (0,1]}$ is tight [BCK]

$(P_\delta)_{\delta \in (0,1]}$ converges as $\delta \to 0$ [HS]

and SLE curves describe the limit object [S, LSW].

- $T$: UST of $\mathbb{Z}^2$
- $d$: intrinsic metric on $T$
- $\mu$: counting measure on $T$
- $\phi: T \rightarrow \mathbb{R}^2$
- $\kappa = 5/4$ is the growth exponent of the 2D LERW

[Schramm '00, Lawler-Schramm-Werner '04, Barlow-Croydon-Kumagai '17, Holden-Sun '18]
Scaling limits of USTs in 3D

Theorem (Angel-Croydon-H.T.-Shiraishi, '20+)

- $U$: UST of $\mathbb{Z}^3$
- $d$: intrinsic metric on $U$
- $\mu$: counting measure on $U$
- $\phi: U \to \mathbb{R}^3$
- $\beta$ is the growth exponent of the 3D LERW.

Let $\mathbb{P}_\delta$ be the law of the measured spatial tree

$$(U, \delta^\beta d, \delta^3 \mu, \delta \phi; \rho),$$

then

- $(\mathbb{P}_\delta)_{\delta \in (0,1]}$ is tight
- $(\mathbb{P}_{2^{-n}})_{n \in \mathbb{N}}$ converges.
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Proof strategy. Use loop-erased random walks.

- Wilson's algorithm rooted at infinity
  [Benjamini-Lyons-Peres-Schramm, '01]

- Scaling limit of loop-erased random walk
  [Kozma, '07], [Li-Shiraishi, '18+]

- With high probability, the LERW has good path properties
  [Sapozhnikov-Shiraishi, '18]
Loop-erased random walks

- Relation to UST
- Properties
Loop-erased random walk (LERW)

$S = [v_1, v_2, ..., v_n]$: simple random walk on $G$.

$\gamma = LE(S)$: the loop-erased random walk (LERW) [Lawler, '80] is the path created by deleting the loops of $S$ in chronological order.
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Wilson's algorithm rooted at infinity

1. \( V = \{x_1, x_2, \ldots, x_N\} \)

2. \( T_0 = \emptyset \)

3. \( \gamma_i \) is the LERW starting at \( x_{i+1} \). Stop \( \gamma_i \) when it first hits a vertex in \( T_i \); otherwise \( \gamma_i \) is an infinite LERW.

4. \( T_{i+1} = T_i \cup \gamma_i \)
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[Benjamini-Lyons-Peres-Schramm]
\( T_N \) has the distribution of the subtree of UST (or USF) on \( \mathbb{Z}^d \) spanned by \( V \).
Wilson’s method rooted at infinity

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3. \( \gamma_i \) is the LERW starting at \( x_{i+1} \). Stop \( \gamma_i \) when it first hits a vertex in \( T_i \); otherwise \( \gamma_i \) is an infinite LERW.

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[Benjamini-Lyons-Peres-Schramm]

Wilson’s method rooted at infinity samples the UST (or USF) on \( \mathbb{Z}^d \).
Towards the scaling limit:
LERW as a curve

- $S$ simple random walk
- $\gamma = LE(S) = [w_0, \ldots, w_T]$ is a path
- $\gamma: [0,T] \to \mathbb{R}^d$ by interpolation
Towards the scaling limit: LERW as a curve

- $S$ simple random walk
- $γ = LE(S) = [w_0, ..., w_T]$ is a path
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Metrics for the space of finite curves
- Hausdorff metric (shape of traces)
- Metric for parameterized curves
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LERW as a curve

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Metrics for the space of finite curves
- Hausdorff metric (shape of traces)
- Metric for parameterized curves
Growth exponent of the LERW

$M_n$: the number of steps that the LERW $\gamma$ takes to exit a ball $B(0,n)$

$E(M_n) \approx n^\beta$
Growth exponent of the LERW

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$E(M_n) \approx n^\beta$

Kenyon, ’00

The growth exponent of the LERW on $\mathbb{Z}^2$ exists and equals

$\kappa = 5/4$. 

LERW on $\mathbb{Z}^2$ (image: Adrien Kassel)
Growth exponent of the LERW

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The growth exponent of the LERW on $\mathbb{Z}^2$ exists and equals

$\kappa = 5/4$.

[Shiraishi, ’18]

The growth exponent of the LERW on $\mathbb{Z}^3$ exists.

$1 < \beta \leq 5/3$ [Lawler, ’99]

$\beta \approx 1.624$ [Wilson, ’10]
Scaling limit of the 2D LERW

[Lawler-Schramm-Werner ’04; Lawler-Viklund ’16+]

The scaling limit of the LERW on $\mathbb{Z}^2$ is SLE(2)

- unparameterized paths [LSW]
- parameterized curves [LV]

Planar graphs + convergence SRW to BM
[Yadin-Yehudayoff, ’11]
Scaling limit of the 3D LERW

[Kozma, '07]

Let $P \subset \mathbb{R}^3$ be a polyhedron.

$P_{2^n} = P \cap 2^{-n}\mathbb{Z}^3$

$\gamma_{2^n}$ LERW on $P_{2^n}$.

Then the law of $\gamma_{2^n}$ converges weakly w.r.t. the Hausdorff topology.
Scaling limit of the 3D LERW

[Li-Shiraishi, '18+]

- Let $B \subset \mathbb{R}^3$ be a ball.
- $B_{2^n} = B \cap 2^{-n} \mathbb{Z}^3$.
- $\beta$ is the growth exponent of the 3D LERW.

\[ \bar{\gamma}_{2^n}(t) = \gamma_{2^n}(2^{\beta n} t) \]

Then the law of $\bar{\gamma}_{2^n}$ converges weakly w.r.t the uniform convergence topology.
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[Li-Shiraishi, '18+]

Let $P \subset \mathbb{R}^3$ be a polyhedron.

$P_{2^n} = P \cap 2^{-n} \mathbb{Z}^3$

$\beta$ is the growth exponent of the 3D LERW.

$\bar{y}_{2^n}(t) = \gamma_{2^n}(2^{\beta n} t)$

Then the law of $\bar{y}_{2^n}$ converges weakly w.r.t. the uniform convergence topology.
Scaling limits of USTs in 3D

Theorem (Angel-Croydon-H.T.-Shiraishi, ‘20+)

Let \( P_\delta \) be the law of the measured spatial tree \((U, \delta^\beta d, \delta^3 \mu, \delta \phi)\).

- \((P_\delta)_{\delta \in (0,1]}\) is tight \(-\) criterion \([BCK]\)  
- \((P_{2^{-m}})_{m \in \mathbb{N}}\) converges \(-\) convergence of parameterized subtrees.

Proof strategy. Use loop-erased random walks.

- Wilson’s algorithm rooted at infinity  
  \([Benjamini-Lyons-Peres-Schramm, ‘01]\)
- Scaling limit of loop-erased random walk  
  \([Kozma, ‘07]\), \([Li-Shiraishi, ‘18+]\)
- With high probability, the LERW has good path properties  
  \([Sapozhnikov-Shiraishi, ‘18]\)
Hittability

\( \gamma: \text{LERW on } \mathbb{Z}^3 \)
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escape event
Hittability

- $\gamma$: LERW on $\mathbb{Z}^3$
- [Sapozhnikov-Shiraishi, '18]
- There exist constants $C, \eta$
- for any $\varepsilon \in (0,1), n \in \mathbb{N}$

there exists

$$P \left( x \in B(n) \text{ with dist}(x, \gamma) \leq \varepsilon n \text{ and } P^x(\text{RW escapes } \gamma) > \varepsilon^n \right) \leq C\varepsilon$$

escape event
Scaling limits of USTs in 3D

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The intrinsic ball
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Proposition

For all $\lambda \geq 1$ and $\delta \in (0,1)$

$$B_{\mathcal{U}}(\lambda^{-1}\delta^{-\beta}) \subset B_E(\delta^{-1}) \text{ with high prob. (i)}$$
The intrinsic ball

Proposition

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$$B_{\mathcal{U}}(\lambda^{-1}\delta^{-\beta}) \subset B_E(\delta^{-1}) \text{ with high prob. } (\lambda)$$

[Shiraishi '18, Li-Shiraishi '19]

$M_n$: the number of steps that the LERW takes to exit a ball $B(0,n)$.

For all $\lambda \geq 1$ and $n \in \mathbb{N}$;

$$M_n \geq \lambda^{-1} n^\beta \text{ with high prob. } (\lambda)$$
The intrinsic ball

**Proposition**

For all \( \lambda \geq 1 \) and \( \delta \in (0,1) \)

\[
B_{\mathcal{U}}(\lambda^{-1}\delta^{-\beta}) \subset B_E(\delta^{-1}) \text{ with high prob. (\( \lambda \))}
\]

**Sketch of proof.**

On \( \delta \mathbb{Z}^3 \):

\[
B_E(1) \quad \partial B(1)
\]
The intrinsic ball

Proposition

For all $\lambda \geq 1$ and $\delta \in (0,1)$

$B_{2\lambda}(\lambda^{-1}\delta^{-\beta}) \subset B_{E}(\delta^{-1})$ with high prob. ($\lambda$)

Sketch of proof.

On $\delta \mathbb{Z}^3$:

$A_k = B(1) \setminus B \left(1 - (2k)^{-1}\right)$

$D_k, \quad \epsilon_k = \lambda^{-c}2^{-k}$

$k = 1, \ldots, k_0$
The intrinsic ball

Proposition

For all $\lambda \geq 1$ and $\delta \in (0,1)$

$$B_\mathcal{U}(\lambda^{-1}\delta^{-\beta}) \subset B_E(\delta^{-1})$$

with high prob. $(\lambda)$

Sketch of proof.

On $\delta\mathbb{Z}^3$:

- $B_E(1)$$
- $D_1, \quad \epsilon_1 = \lambda^{-c}2^{-1}$
- $A_1 = B(1) \setminus B(1/2)$
- $\partial B(1)$
The intrinsic ball

**Proposition**

For all $\lambda \geq 1$ and $\delta \in (0,1)$

$$B_{\mathcal{U}}(\lambda^{-1}\delta^{-\beta}) \subset B_E(\delta^{-1}) \text{ with high prob. (1)}$$

**Sketch of proof.**

On $\delta \mathbb{Z}^3$:

- $A_1 = B(1) \setminus B(1/2)$
- $D_1, \quad \epsilon_1 = \lambda^{-c}2^{-1}$
The intrinsic ball

Proposition

For all $\lambda \geq 1$ and $\delta \in (0,1)$

$$B_{\mathcal{U}}(\lambda^{-1} \delta^{-\beta}) \subset B_{E}(\delta^{-1})$$

with high prob. ($\lambda$)

Sketch of proof.

On $\delta \mathbb{Z}^3$:

- $A_1 = B(1) \setminus B(1/2)$
- $D_1, \quad \epsilon_1 = \lambda^{-c} 2^{-1}$

The intrinsic ball

Proposition

For all $\lambda \geq 1$ and $\delta \in (0,1)$

$$B_{\mathcal{U}}(\lambda^{-1} \delta^{-\beta}) \subset B_{E}(\delta^{-1})$$

with high prob. ($\lambda$)

Sketch of proof.

On $\delta \mathbb{Z}^3$:

- $A_1 = B(1) \setminus B(1/2)$
- $D_1, \quad \epsilon_1 = \lambda^{-c} 2^{-1}$
The intrinsic ball

Proposition

For all $\lambda \geq 1$ and $\delta \in (0,1)$

$$B_{\|\cdot\|}(\lambda^{-1}\delta^{-\beta}) \subset B_E(\delta^{-1})$$

with high prob. ($\lambda$)

Sketch of proof.

On $\delta \mathbb{Z}^3$:

$$A_k = B(1) \setminus B \left(1 - (2k)^{-1}\right)$$

$$D_k, \quad \epsilon_k = \lambda^{-c}2^{-k}$$

$$k = 1, \ldots, k_0$$
The intrinsic ball

Proposition

For all $\lambda \geq 1$ and $\delta \in (0,1)$

$$B_E(\lambda^{-1}\delta^{-1}) \subset B_{\infty}(\delta^{-\beta}) \subset B_E(\lambda \delta^{-1})$$

with high prob. ($\lambda$)

Sketch of proof.

On $\delta \mathbb{Z}^3$:
The intrinsic ball

Proposition

For all $\lambda \geq 1$ and $\delta \in (0,1)$

$$B_E(\lambda^{-1}\delta^{-1}) \subset B_\mathcal{U}(\delta^{-\beta}) \subset B_E(\lambda\delta^{-1}) \text{ with high prob. } (\lambda)$$

Theorem

$$\lambda^{-1}R^3 \leq \mu_\mathcal{U}(B_\mathcal{U}(R^\beta)) \leq \lambda R^3 \text{ with high prob. } (\lambda)$$
The intrinsic ball

**Proposition**

For all $\lambda \geq 1$ and $\delta \in (0,1)$

$$B_E(\lambda^{-1}\delta^{-1}) \subset B_\mathcal{U}(\delta^{-\beta}) \subset B_E(\lambda\delta^{-1})$$

with high prob. ($\lambda$)

**Theorem**

$$\lambda^{-1}R^{3/\beta} \leq \mu_\mathcal{U}(B_\mathcal{U}(R)) \leq \lambda R^{3/\beta}$$

with high prob. ($\lambda$)
Properties of 3D scaling limits

Theorem (Angel-Croydon-H.T.-Shiraishi ’20+)

$(\mathcal{U}, d_\mathcal{U}, \phi_\mathcal{U})$: sub-sequential limit of the UST on $\mathbb{Z}^3$

- $(\mathcal{U}, d_\mathcal{U})$ is a metric space, $\phi_\mathcal{U} : \mathcal{U} \to \mathbb{R}^3$,
- complete, locally compact,
- $\mathbb{R}$-tree.

- We have a unique isometry $\varphi : \mathbb{R}_+ \to (\mathcal{U}, d_\mathcal{U})$ (an end) with $\varphi(0) = 0$.

- The Hausdorff dimension is

$$\dim \mathcal{U} = \frac{3}{\beta}.$$
Thank you!